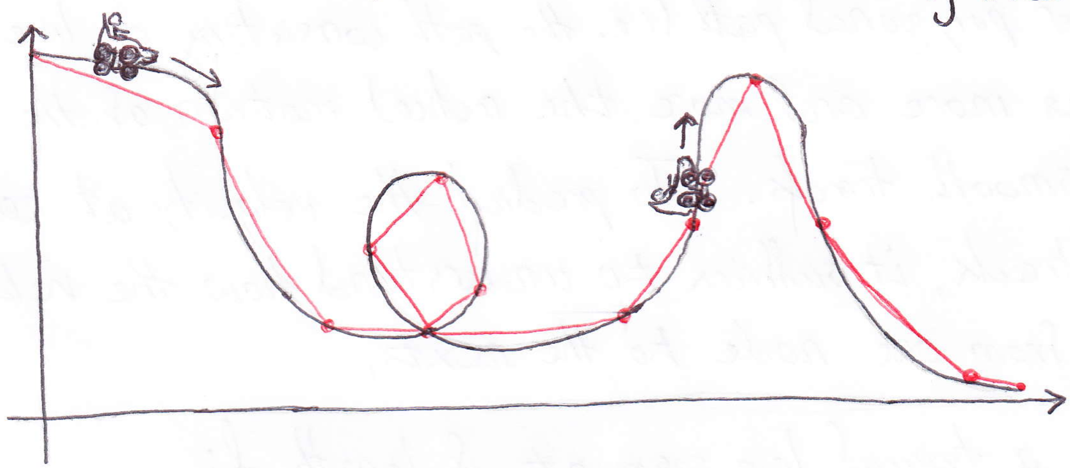
(1 . 2)

Inner Product

Suppose that you are considering to ride a gravity-driven roller coaster that is securely attached to a smooth, frictionless track. Perhaps the ride looks something like this:



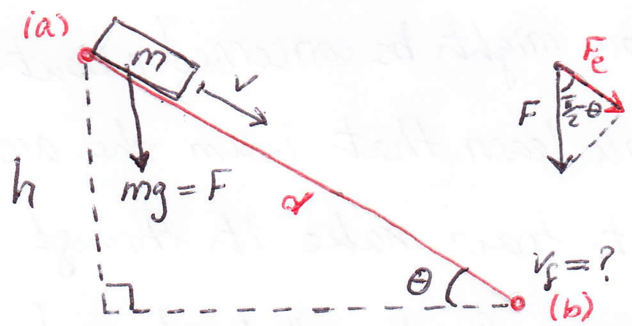
Before getting on, you might be concerned about a few questionss (Especially when you learn that I am the architect)

1. Will the first train make it through the mountain?
2. Will the train in front roll back and collide with the one behind it?
3. Will the rider survive the accelleration?

To help answer these questions, it would be nice to know the velocity at each point in the roller coaster's path.

The assumption that the path is smooth allows us to simplify things a little bit. We can approximate our path with tiny line segments. As the length of each of these segments gets smaller and smaller, the motion of the roller-coaster train over polygonal path (i.e. the path consisting of line segments) resembles more and more the actual motion of the train on the smooth track. To predict the velocity at each point on the track, it suffices to understand how the velocity changes from one node to the next.

Consider a typical line segment of length d:



If the mass of the train is m, the force acting on it is $F = mg$. This force is directed downward along an axis perpendicular to the horizontal. Since the train is securely attached to the track, it can only be accelerated in a direction that is parallel to the track; The force perpendicular to the track is wasted. Hence the force that causes acceleration is given by $F_e = F\cos(\frac{\pi}{2}-\theta) = F\sin\theta$, where θ is the angle between the line segment and the horizontal.

Let v be the train's velocity at node (a), v_f - the train's velocity at node (b), and t be the time it takes for the train to get from point (a) to point (b). Then, by basic laws of motion

$$F_e = m a_e;$$

$$v_f = v + t a_e;$$

$$d = vt + \frac{1}{2} a_e t^2.$$

Hence

$$a_e = \frac{F_e}{m} = \frac{mg\sin\theta}{m} = g\sin\theta;$$

$$t = \frac{v_f - v}{a_e};$$

$$d = v\left(\frac{v_f - v}{a_e}\right) + \frac{1}{2} a_e \left(\frac{v_f - v}{a_e}\right)^2$$

$$= \frac{2vv_f - 2v^2 + v_f^2 - 2vv_f + v^2}{2a_e}$$

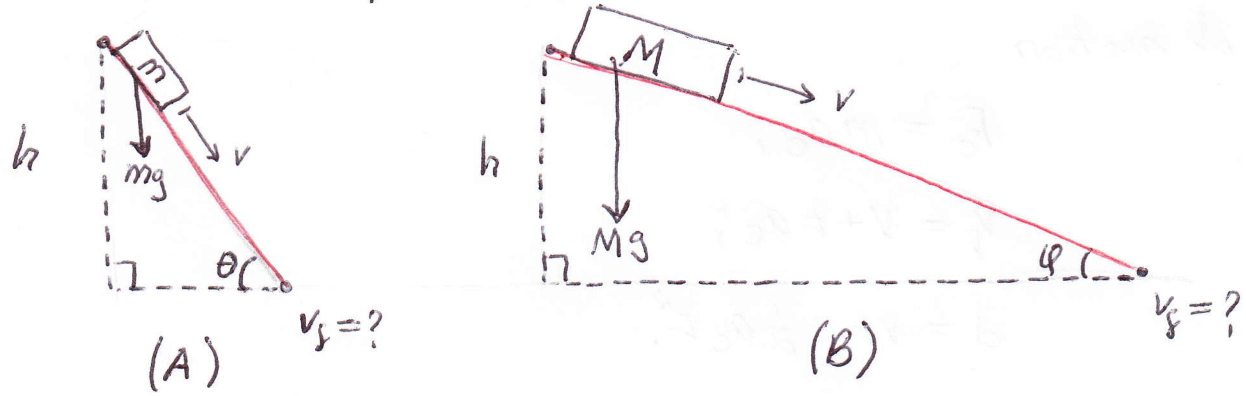
$$= \frac{v_f^2 - v^2}{2a_e}$$

It follows that $v_f = \sqrt{v^2 + 2a_e d} = \sqrt{v^2 + 2g\sin\theta\, d}$

We can drop the dependency on θ by noting that $\sin\theta = \frac{h}{d}$ where h is the vertical displacement from node (a) to node (b). Then $d = \frac{h}{\sin\theta}$ and $v_f = \sqrt{v^2 + 2gh}$

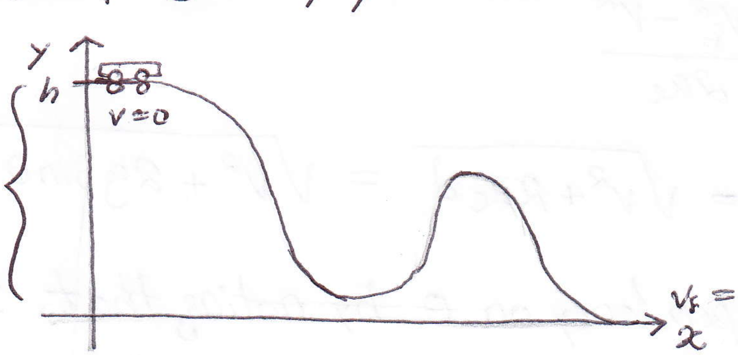
The last equation tells us that the velocity at node (b) is a function of the velocity at node (a) and displacement h. Here is what it means:

Ex. Calculate the final speed of the train for both pictures. Where is the final speed faster?

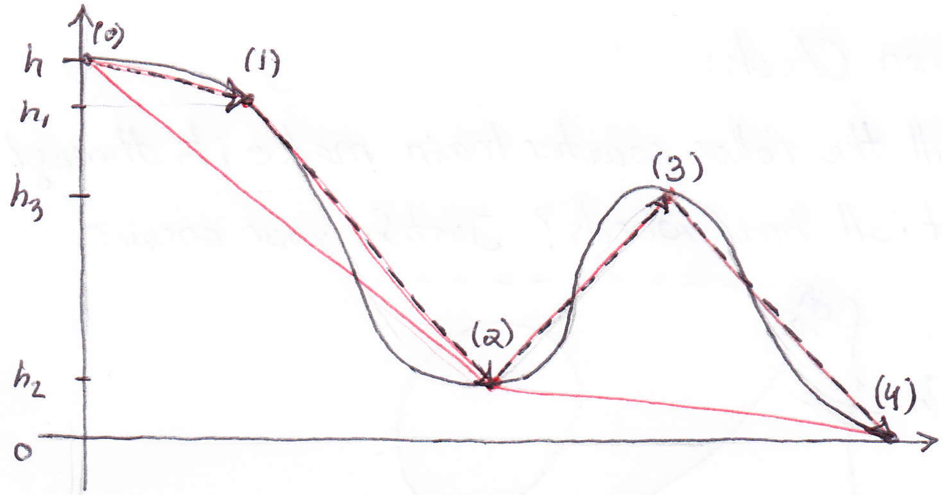


Solution: The final speed depends entirely on the initial velocity v and displacement h. The final speed in (A) and in (B) is given by $v_f = \sqrt{v^2 + 2gh}$ despite differences in mass, length of pathway, and steepness of the pathway.

Ex. Calculate the final speed of the roller coaster in the picture below. Justify your answer.



Solution: The final speed $v_f = \sqrt{2gh}$. To prove this, we will demonstrate that the final speed of the roller coaster train along any polygonal path with beginning and end points matching the actual path is given by $v_f = \sqrt{2gh}$. The illustration below should suffice to give you an idea of what happens.



First concentrate on the red line with nodes (0), (2), and (4)

Observe that

$$v_2 = \sqrt{0^2 + 2g(h-h_2)}$$

$$v_4 = \sqrt{v_2^2 + 2g(h_2 - 0)} = \sqrt{2g(h-h_2) + 2gh_2} = \sqrt{2gh}$$

Doing the same computation for the refined path with nodes (0), (1), (2), (3), and (4) we obtain

$$v_1 = \sqrt{0^2 + 2g(h-h_1)}$$

$$v_2 = \sqrt{2g(h-h_1) + 2g(h_1 - h_2)} = \sqrt{2g(h-h_2)}$$

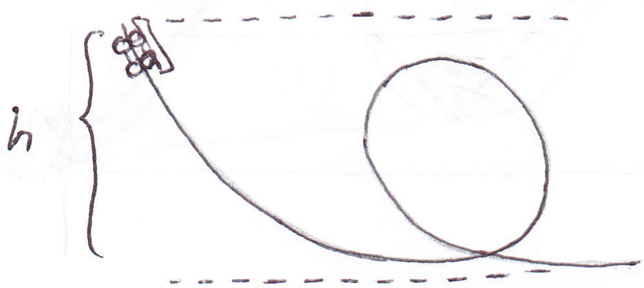
$$v_3 = \sqrt{2g(h-h_2) + 2g(h_2 - h_3)} = \sqrt{2g(h-h_3)}$$

$$v_4 = \sqrt{2g(h-h_3) + 2g(h_3 - 0)} = \sqrt{2gh}$$

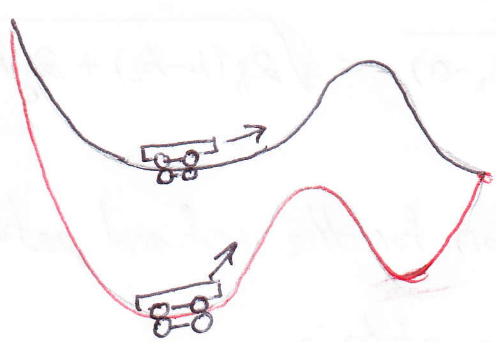
It is clear that as we refine our polygonal curve, increasing the number of nodes that match the actual path, the value at the endnode will always remain $\sqrt{2gh}$, which proves our hypothesis about the final speed of the train.

Comprehension Check:

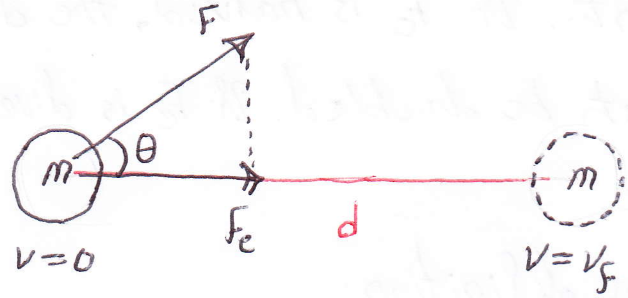
(a) Will the roller coaster train make it through the loop or will it roll backwards? Justify your answer.



(b) Is there any difference between the final speeds of trains over the top and bottom tracks?



Many heighly important concepts in physics and in mathematics depend on our understanding the precise nature of the motion of a particle along a path under the action of a force field that causes the motion. Suppose that a bead of mass m is accelerated from rest to a speed v_f along a rigid, frictionless wire by a force of magnitude F. In the process, the bead will have displaced a distance d from its initial position just as it reaches the speed v_f.



What might be the value of d in terms of m, F, and v_f?

Let θ be the angle between the force vector and the path. Then $F_e = F\cos\theta$ is the portion of the force that is actually responsible for the acceleration. If t is the time it takes the bead to displace a distance d, we get the following equations:

$$m a_e = F_e = F\cos\theta;$$

$$a_e t = v_f;$$

$$\frac{1}{2} a_e t^2 = d.$$

Hence,

$$a_e = \frac{F\cos\theta}{m};$$

$$t = \frac{v_f}{a_e};$$

$$\frac{1}{2}\,\frac{F\cos\theta}{m}\,\frac{v_f^2}{(F\cos\theta)^2}\cdot m^2 = d$$

In particular, $\frac{1}{2} m v_f^2 = F \cos\theta \cdot d. = F_e \cdot d$

The formula above tells us that the distance required to accelerate a particle from rest to a fixed speed v_f is inversely proportional to the force component F_e, acting in the direction of motion.

$$d \propto \frac{1}{F_e}$$

That is, $F_e \cdot d$ is a constant. If F_e is halved, the distance through which F_e acts must be doubled. If F_e is divided by 3, d must be trippled, etc.

This motivates the following definition:

Def: Suppose that a force vector $\vec{F}$ accelerates a particle through a displacement $\vec{d}$. If the angle between the vectors $\vec{F}$ and $\vec{d}$ is θ, then the work, W, done by $\vec{F}$ on the particle is the quantity $W = \|\vec{F}\|\|\vec{d}\| \cos\theta$.

The quantity W is a product of two vectors. Beside its physical interpritation, expressions of the form W have many useful applications within mathematics and other sciences.

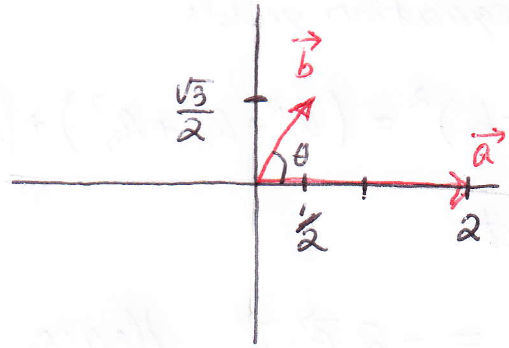
Def: The dot product (inner product) of two vectors $\vec{a}$ and $\vec{b}$ is the scalar given by

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. We agree that $\vec{a} \cdot \vec{0} = 0$

Ex. Find the angle between $\vec{a} = 2\vec{i}$ and $\vec{b} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$ and compute the inner product.

Solution:



$$\tan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}. \quad \text{Hence } \theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3} \text{ or } 60^\circ$$

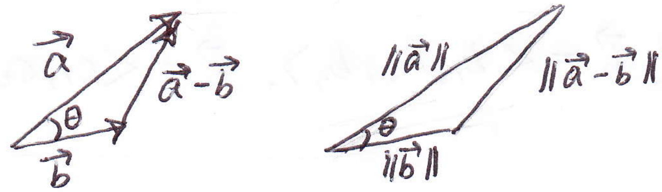
$$\vec{a}\cdot\vec{b} = \|2i\|\,\|\tfrac{1}{2}i + \tfrac{\sqrt{3}}{2}j\|\cos\left(\tfrac{\pi}{3}\right) = 2\cdot 1\cdot\tfrac{1}{2} = 1.$$

Computing the angle between two vectors to then evaluate the dot product can be rather challenging. There is a simpler way!

Thm: If $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, then

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Proof: This is just an application of the familiar law of cosines.



By law of cosines, $\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta =$
$= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a}\cdot\vec{b}$.

Expanding the above equation yields

$$(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2 = (a_1^2+a_2^2+a_3^2) + (b_1^2+b_2^2+b_3^2) - 2\vec{a}\cdot\vec{b}$$

Upon simplifying, we get

$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\vec{a}\cdot\vec{b}. \text{ Hence}$$

$$a_1b_1 + a_2b_2 + a_3b_3 = \vec{a}\cdot\vec{b} \text{ as desired.}$$

Ex. If $\vec{a} = 2\vec{i} - \vec{j} + 3\vec{k}$ & $\vec{b} = 6\vec{i} + 7\vec{j} + \vec{k}$,

$$\vec{a}\cdot\vec{b} = 2\cdot 6 + (-1)\cdot 7 + 3\cdot 1 = 8$$

The dot product has many of the same properties as ordinary multiplication:

Thm: For any vectors $\vec{a}, \vec{b}, \vec{c}$, and any scaler p

1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
2. $(p\vec{a})\cdot\vec{b} = \vec{a}\cdot(p\vec{b}) = p(\vec{a}\cdot\vec{b})$
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

Proof: We'll prove 3 and leave the rest as an exercise.

Set $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, $\vec{c} = \langle c_1, c_2, c_3\rangle$.

Then

$\vec{a}\cdot(\vec{b}+\vec{c}) = a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3) =$

$(a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.

Observe that $\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}}$ and that $\vec{a}\cdot\vec{b} = 0$ if and only if (iff) $\vec{a}$ and $\vec{b}$ are orthogonal.

Finding the angle between two vectors

Let θ be the smallest angle between $\vec{a}$ and $\vec{b}$. Then $0 \le \theta \le \pi$ and $\|\vec{a}\|\,\|\vec{b}\| \cos\theta = \vec{a}\cdot\vec{b}$

Hence $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$ and $\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$.

Thus, if $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$,

$$\theta = \cos^{-1}\left(\frac{\sum_{r=1}^{3} a_r b_r}{\|\vec{a}\|\,\|\vec{b}\|}\right).$$

Ex. Find the angle between the vectors $\vec{i}+\vec{j}+\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$.

Solution: $\|\vec{i}+\vec{j}+\vec{k}\| = \sqrt{1+1+1} = \sqrt{1+1+(-1)^2} = \|\vec{i}+\vec{j}-\vec{k}\|$

and $(\vec{i}+\vec{j}+\vec{k})\cdot(\vec{i}+\vec{j}-\vec{k}) = 1+1-1 = 1$. Hence

$\cos\theta = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$, $\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.23$ rad or $71°$.

The Cauchy-Schwarz Inequality

Thm: For any two vectors $\vec{a}, \vec{b}$, we have

$$|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

with equality iff $\vec{a}$ is a scalar multiple of $\vec{b}$, or one of them is $\vec{0}$.

Proof: $|\vec{a}\cdot\vec{b}| = \left| \|\vec{a}\| \|\vec{b}\| \cos\theta \right| = \|\vec{a}\| \|\vec{b}\| |\cos\theta| \leq \|\vec{a}\| \|\vec{b}\|$
with equality iff $|\cos\theta| = 1$, iff $\cos\theta = \pm 1$ iff $\theta = 0$ or $\theta = \pi$.
In that case $\vec{a}$ is a scalar multiple of $\vec{b}$ (Why?)

Ex. Verify the Cauchy-Schwarz inequality for $\vec{a} = -\vec{i} + \vec{j} + \vec{k}$ and $\vec{b} = 3\vec{i} + \vec{k}$.

Solution: $|\vec{a}\cdot\vec{b}| = |-3+0+1| = 2$, while

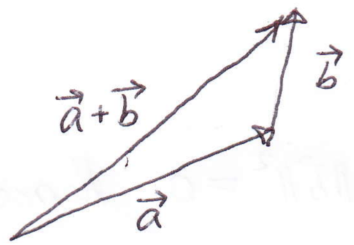
$$\|\vec{a}\| \|\vec{b}\| = \sqrt{1+1+1}\,\sqrt{9+1} = \sqrt{3}\,\sqrt{10} > \sqrt{3}\,\sqrt{3} = 3 > 2.$$

Hence $|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$ as desired.

A useful consequence of the Cauchy-Schwarz thm. is the triangle inequality, which states that the shortest distance between two points is a line.

Thm: Let $\vec{a}, \vec{b}$ be vectors, then $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Proof:



Notice that $\|\vec{a}+\vec{b}\| = \sqrt{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})}$

Hence,

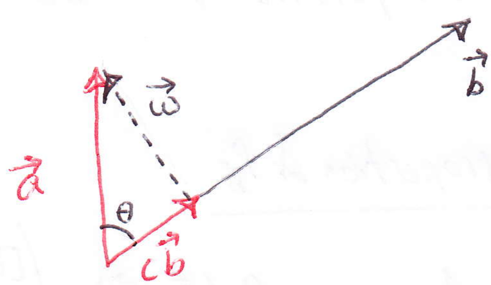
$$\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$

$$\leq \|\vec{a}\|^2 + 2|\vec{a}\cdot\vec{b}| + \|\vec{b}\|^2 \leq \|\vec{a}\|^2 + 2\|\vec{a}\|\,\|\vec{b}\| + \|\vec{b}\|^2 = (\|\vec{a}\| + \|\vec{b}\|)^2$$

It follows that $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ (upon taking roots)

Orthogonal Projection

Recall the motivation behind the dot product. Given vectors $\vec{a}, \vec{b}$. How do we determine that part of $\vec{a}$ that "acts" in the direction of $\vec{b}$?



$\vec{\omega}\cdot\vec{b} = 0$ iff
$\vec{\omega} \perp \vec{b}$

Observe that $\vec{a}$ can be written as $c\vec{b}+\vec{\omega}$ where c is some scalar to be determined and $\vec{\omega}$ is perpendicular to $\vec{b}$.

Then we can try to solve for c using the equations

$$\vec{w} = \vec{a} - c\vec{b}$$

$$\vec{w} \cdot \vec{b} = 0$$

which yields $(\vec{a} - c\vec{b}) \cdot \vec{b} = \vec{a} \cdot \vec{b} - c\|\vec{b}\|^2 = 0$. Hence,
$c = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}$ when $\|\vec{b}\| \neq 0$ and $c = 0$ otherwise.

Alternatively, observe that the component of $\vec{a}$ acting along $\vec{b}$ has magnitude $\|\vec{a}\| \cos\theta$ and direction $\dfrac{\vec{b}}{\|\vec{b}\|}$. Hence the vector acting on the line generated by $\vec{b}$ is $(\|\vec{a}\| \cos\theta) \dfrac{\vec{b}}{\|\vec{b}\|} =$

$$= \left(\frac{\|\vec{a}\| \, \|\vec{b}\| \cos\theta}{\|\vec{b}\|} \right) \frac{\vec{b}}{\|\vec{b}\|} = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} \right) \frac{\vec{b}}{\|\vec{b}\|} = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$$

The result above leads us to define a very important function.

Def: Let V be a collection of vectors. Fix $\vec{b} \in V$ and define $P_{\vec{b}} : V \mapsto V$ by $P_{\vec{b}}(\vec{v}) = \left(\dfrac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$. Then $P_{\vec{b}}$ is the projection function that computes the component of $\vec{v}$ that acts along the line generated by $\vec{b}$.

<u>Important properties of $P_{\vec{b}}$</u>

1) $P_{\vec{b}}(\vec{a} + \vec{c}) = P_{\vec{b}}(\vec{a}) + P_{\vec{b}}(\vec{c})$ because $P_{\vec{b}}(\vec{a} + \vec{c}) = \left(\dfrac{[\vec{a} + \vec{c}] \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b}$

$$= \frac{\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} + \frac{\vec{c} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = P_{\vec{b}}(\vec{a}) + P_{\vec{b}}(\vec{c})$$

2) $\forall\, k \in \mathbb{R}$, $P_{\vec{b}}(k\vec{a}) = k P_{\vec{b}}(\vec{a})$ because $P_{\vec{b}}(k\vec{a}) = \left(\frac{[k\vec{a}]\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$
$= k\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b} = k P_{\vec{b}}(\vec{a})$

These two properties imply that projections belong to a very important class of functions called "Linear maps", which are studied extensively in a later course (Linear Algebra).
We will put these functions to good use in the next chapter.

Ex. Find the orthogonal projection of $\vec{i}+\vec{j}$ on $\vec{i}-2\vec{j}$.

Solution:

$$P_{i-2j}(\vec{i}+\vec{j}) = \left(\frac{<1,1>\cdot<1,-2>}{\|<1,-2>\|^2}\right)(\vec{i}-2\vec{j}) = \frac{-1}{5}(\vec{i}-2\vec{j})$$